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L_p Markov–Bernstein Inequalities on Arcs of the Circle

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Let $0 < p < \infty$ and $0 \leqslant \alpha < \beta \leqslant 2\pi$. We prove that for trigonometric polynomials s_n of degree $\leqslant n$, we have

$$\int_\alpha^\beta |s_n'(\theta)|^p \left[\left| \sin\left(\frac{\theta-\alpha}{2}\right) \right| \left| \sin\left(\frac{\theta-\beta}{2}\right) \right| + \left(\frac{\beta-\alpha}{n}\right)^2 \right]^{p/2} d\theta$$

$$\leqslant cn^p \int_\alpha^\beta |s_n(\theta)|^p \, d\theta,$$

where c is independent of α, β, n, s_n. The essential feature is the uniformity in α and β of the estimate. The result may be viewed as an L_p form of Videnskii's inequalities. © 2001 Academic Press

1. INTRODUCTION AND RESULTS

The classical Markov inequality for trigonometric polynomials

$$s_n(\theta) := \sum_{j=0}^n (c_j \cos j\theta + d_j \sin j\theta)$$

of degree $\leqslant n$ is

$$\|s_n'\|_{L_\infty[0,\, 2\pi]} \leqslant n \, \|s_n\|_{L_\infty[0,\, 2\pi]}.$$

The same factor n occurs in the L_p analogue. See [1] or [3]. In the 1950s V. S. Videnskii generalized the L_∞ inequality to the case where the interval over which the norm is taken is shorter than the period. An accessible reference discussing this is the book of Borwein and Erdelyi

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[1, pp. 242–245]. We formulate this in the symmetric case: let $0<\omega<\pi$. Then there is the sharp inequality

$$|s_n'(\theta)| \left[1-\frac{\cos^2 \omega/2}{\cos^2 \theta/2}\right]^{1/2} \leqslant n \, \|s_n\|_{L_\infty[-\omega,\omega]}, \qquad \theta\in[-\omega,\omega].$$

This implies that

$$\sup_{\theta\in[-\pi,\pi]} |s_n'(\theta)| \left[\left|\sin\left(\frac{\theta-\omega}{2}\right)\right| \left|\sin\left(\frac{\theta+\omega}{2}\right)\right|\right]^{1/2} \leqslant n \, \|s_n\|_{L_\infty[-\omega,\omega]}$$

and for $n\geqslant n_0(\omega)$, this gives rise to the sharp Markov inequality

$$\|s_n'\|_{L_\infty[-\omega,\omega]} \leqslant 2n^2 \cot\frac{\omega}{2} \, \|s_n\|_{L_\infty[-\omega,\omega]}.$$

What are the L_p analogues? This question arose originally in connection with large sieve inequalities [7], on subarcs of the circle. Here we prove:

THEOREM 1.1. *Let $0<p<\infty$ and $0\leqslant\alpha<\beta\leqslant 2\pi$. Then for trigonometric s_n of degree $\leqslant n$,*

$$\int_\alpha^\beta |s_n'(\theta)|^p \left[\left|\sin\left(\frac{\theta-\alpha}{2}\right)\right| \left|\sin\left(\frac{\theta-\beta}{2}\right)\right| + \left(\frac{\beta-\alpha}{n}\right)^2\right]^{p/2} d\theta$$
$$\leqslant Cn^p \int_\alpha^\beta |s_n(\theta)|^p \, d\theta. \tag{1}$$

Here C is independent of α, β, n, s_n.

This inequality confirms a conjecture of Erdelyi [4]. We deduce Theorem 1.1 from an analogous inequality for algebraic polynomials:

THEOREM 1.2. *Let $0<p<\infty$ and $0\leqslant\alpha<\beta\leqslant 2\pi$. Let*

$$\varepsilon_n(z) := \frac{1}{n}\left[|z-e^{i\alpha}|\,|z-e^{i\beta}| + \left(\frac{\beta-\alpha}{n}\right)^2\right]^{1/2}. \tag{2}$$

Then for algebraic polynomials P of degree $\leqslant n$,

$$\int_\alpha^\beta |(P'\varepsilon_n)(e^{i\theta})|^p \, d\theta \leqslant C \int_\alpha^\beta |P(e^{i\theta})|^p \, d\theta. \tag{3}$$

Here C is independent of α, β, n, s_n.

Our method of proof uses Carleson measures much as in [8, 9], but also uses ideas from [7] where large sieve inequalities were proved for subarcs of the circle. We could also replace pth powers by more general expressions involving convex increasing functions composed with pth powers, provided a result of Carleson on Carleson measures admits a generalisation from L_p spaces to certain Orlicz spaces. We believe that such an extension must be possible, but have not been able to find it in the literature. So we restrict ourselves to L_p estimates.

We shall prove Theorem 1.2 in Section 2, deferring some technical estimates. In Section 3, we present estimates involving the function ε and also estimate the norms of certain Carleson measures. In Section 4, we prove Theorem 1.1.

2. THE PROOF OF THEOREM 1.2.

Throughout, $C, C_0, C_1, C_2, \ldots$ denote constants that are independent of α, β, n and polynomials P of degree $\leqslant n$ or trigonometric polynomials s_n of degree $\leqslant n$. They may however depend on p. The same symbol does not necessarily denote the same constant in different occurrences. We shall prove Theorem 1.2 in several steps:

(I) *Reduction to the Case* $0 < \alpha < \pi$; $\beta := 2\pi - \alpha$

After a rotation of the circle, we may assume that our arc $\{e^{i\theta}\colon \theta \in [\alpha, \beta]\}$ has the form

$$\Delta = \{e^{i\theta}\colon \theta \in [\alpha', 2\pi - \alpha']\},$$

where $0 \leqslant \alpha' < \pi$. Then Δ is symmetric about the real line, and this simplifies use of a conformal map below. Moreover, then

$$\beta - \alpha = 2(\pi - \alpha').$$

Thus, dropping the prime, it suffices to consider $0 < \alpha < \pi$, and $\beta - \alpha$ replaced everywhere by $2(\pi - \alpha)$. Thus in the sequel, we assume that

$$\Delta = \{e^{i\theta}\colon \theta \in [\alpha, 2\pi - \alpha]\}; \tag{4}$$

$$R(z) = (z - e^{i\alpha})(z - e^{-i\alpha}) = z^2 - 2\cos\frac{\alpha}{2}\, z + 1; \tag{5}$$

and (dropping the subscript n from ε_n as well as an inconsequential factor of 2 in ε_n in (2)),

$$\varepsilon(z)=\frac{1}{n}\left[\,|R(z)|+\left(\frac{\pi-\alpha}{n}\right)^2\right]^{1/2}. \tag{6}$$

We can now begin the main part of the proof:

(II) *Pointwise Estimates for $P'(z)$ when $p\geqslant 1$*

By Cauchy's integral formula for derivatives,

$$\begin{aligned}|P'(z)|&=\left|\frac{1}{2\pi i}\int_{|t-z|=\varepsilon(z)/100}\frac{P(t)}{(t-z)^2}\,dt\right|\\&\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}e^{i\theta}\right)\right|d\theta\Big/\left(\frac{\varepsilon(z)}{100}\right).\end{aligned}$$

Then Hölder's inequality gives

$$\begin{aligned}&|P'(z)|\,\varepsilon(z)\leqslant 100\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}e^{i\theta}\right)\right|^p d\theta\right)^{1/p}\\&\Rightarrow(|P'(z)|\,\varepsilon(z))^p\leqslant 100^p\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}e^{i\theta}\right)\right|^p d\theta.\end{aligned}$$

(III) *Pointwise Estimates for $P'(z)$ when $p<1$*

We follow ideas in [9]. Suppose first that P has no zeros inside or on the circle $\gamma:=\{t\colon |t-z|=\frac{\varepsilon(z)}{100}\}$. Then we can choose a single valued branch of P^p there, with the properties

$$\frac{d}{dt}P(t)^p_{|t=z}=pP(z)^p\,\frac{P'(z)}{P(z)}$$

and

$$|P^p(t)|=|P(t)|^p.$$

Then by Cauchy's integral formula for derivatives,

$$\begin{aligned}p\,|P'(z)|\,|P(z)|^{p-1}&=\left|\frac{1}{2\pi i}\int_{|t-z|=\varepsilon(z)/100}\frac{P^p(t)}{(t-z)^2\,dt}\right|\\&\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(z+\frac{\varepsilon(z)}{100}e^{i\theta}\right)\right|^p d\theta\Big/\left(\frac{\varepsilon(z)}{100}\right).\end{aligned}$$

Since also (by Cauchy or by subharmonicity)

$$|P(z)|^p \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta$$

and since $1 - p > 0$, we deduce that

$$p\, |P'(z)|\, \varepsilon(z) \leqslant 100 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta \right)^{1/p}$$

$$\Rightarrow (|P'(z)|\, \varepsilon(z))^p \leqslant \left(\frac{100}{p}\right)^p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta.$$

Now suppose that P has zeros inside γ. We may assume that it does not have zeros on γ (if necessary change $\varepsilon(z)$ a little and then use continuity). Let $B(z)$ be the Blaschke product formed from the zeros of P inside γ. This is the usual Blaschke product for the unit circle, but scaled to γ so that $|B| = 1$ on γ. Then the above argument applied to (P/B) gives

$$(|(P/B)'\,(z)|\, \varepsilon(z))^p \leqslant \left(\frac{100}{p}\right)^p \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta.$$

Moreover, as above

$$|P/B(z)|^p \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta,$$

while Cauchy's estimates give

$$|B'(z)| \leqslant \frac{100}{\varepsilon(z)}.$$

Then these last three estimates give

$$\begin{aligned} |P'(z)|^p\, \varepsilon(z)^p &\leqslant (|(P/B)'\,(z)\, B(z)| + |P/B(z)|\, |B'(z)|)^p\, \varepsilon(z)^p \\ &\leqslant \left\{ \left(\frac{200}{p}\right)^p + 200^p \right\} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta \right]. \end{aligned}$$

In summary, the last two steps give for all $p > 0$,

$$|P'\varepsilon|^p\,(z) \leqslant C_0 \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P\left(z + \frac{\varepsilon(z)}{100} e^{i\theta}\right) \right|^p d\theta, \tag{7}$$

where

$$C_0 := 200^p(1+p^{-p}).$$

(IV) *Integrate the Pointwise Estimates*

We obtain by integration of (7) that

$$\int_\alpha^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \leqslant C_0 \int |P(z)|^p \, d\sigma, \tag{8}$$

where the measure σ is defined by

$$\int f \, d\sigma := \int_\alpha^{2\pi-\alpha} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{is} + \frac{\varepsilon(e^{is})}{100} e^{i\theta} \right) d\theta \right] ds. \tag{9}$$

We now wish to pass from the right-hand side of (9) to an estimate over the whole unit circle. This passage would be permitted by a famous result of Carleson, provided P is analytic off the unit circle, and provided it has suitable behaviour at ∞. To take care of the fact that it does not have the correct behaviour at ∞, we need a conformal map:

(V) *The Conformal Map Ψ of $\mathbb{C}\backslash\Delta$ onto $\{w: |w|>1\}$*

This is given by

$$\Psi(z) = \frac{1}{2\cos\alpha/2}\,[z+1+\sqrt{R(z)}],$$

where the branch of $\sqrt{R(z)}$ is chosen so that it is analytic off Δ and behaves like $z(1+o(1))$ as $z\to\infty$. Note that $\sqrt{R(z)}$ and hence $\Psi(z)$ have well defined boundary values (both non-tangential and tangential) as z approaches Δ from inside or outside the unit circle, except at $z=e^{\pm i\alpha}$. We denote the boundary values from inside by $\sqrt{R(z)}_+$ and $\Psi(z)_+$ and from outside by $\sqrt{R(z)}_-$ and $\Psi(z)_-$. We also set (unless otherwise specified)

$$\Psi(z) := \Psi(z)_+, \qquad z\in\Delta\backslash\{e^{i\alpha}, e^{-i\alpha}\}.$$

See [6] for a detailed discussion and derivation of this conformal map. Let

$$\ell := \text{least positive integer} > \frac{1}{p}. \tag{10}$$

In [7, Lemma 3.2] it was shown that there is a constant C_1 (independent of α, β, n) such that

$$a \in \Delta \qquad \text{and} \quad |z-a| \leqslant \frac{\varepsilon(a)}{100} \Rightarrow |\Psi(z)|^{n+\ell} \leqslant C_1.$$

(There ℓ was replaced by 2, but the proof is the same; the constant C_1 depends on ℓ and so on p). Then we deduce from (8) that

$$\int_{\alpha}^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \leqslant C_1^p C_0 \int \left| \frac{P}{\Psi^{n+\ell}} \right|^p d\sigma. \tag{11}$$

Since the form of Carleson's inequality that we use involves functions analytic defined on the unit ball, we now split σ into its parts with support inside and outside the unit circle: for measurable S, let

$$\begin{aligned} \sigma^+(S) &:= \sigma(S \cap \{z : |z| < 1\}); \\ \sigma^-(S) &:= \sigma(S \cap \{z : |z| > 1\}). \end{aligned} \tag{12}$$

Moreover, we need to "reflect σ^- through the unit circle": let

$$\sigma^{\#}(S) := \sigma^-\left(\frac{1}{S}\right) := \sigma^-\left(\left\{\frac{1}{t} : t \in S\right\}\right). \tag{13}$$

Then since the unit circle Γ has $\sigma(\Gamma) = 0$, (11) becomes

$$\begin{aligned} &\int_{\alpha}^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \\ &\qquad \leqslant C_1^p C_0 \left(\int \left| \frac{P}{\Psi^{n+\ell}} \right|^p (t) \, d\sigma^+(t) + \int \left| \frac{P}{\Psi^{n+\ell}} \right|^p \left(\frac{1}{t}\right) d\sigma^{\#}(t) \right). \end{aligned} \tag{14}$$

We next focus on handling the first integral in the last right-hand side:

(VI) *Estimate the Integral Involving* σ^+

We are now ready to apply Carleson's result. Recall that a positive Borel measure μ with support inside the unit ball is called a *Carleson measure* if there exists $A > 0$ such that for every $0 < h < 1$ and every sector

$$S := \{re^{i\theta} : r \in [1-h, 1]; |\theta - \theta_0| \leqslant h\}$$

we have

$$\mu(S) \leqslant Ah.$$

The smallest such A is called the Carleson norm of μ and denoted $N(\mu)$. See [5] for an introduction. One feature of such a measure is the inequality

$$\int |f|^p \, d\mu \leqslant C_2 N(\mu) \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \tag{15}$$

valid for every function f in the Hardy p space on the unit ball. Here C_2 depends only on p. See [5, pp. 238] and also [5, p. 31; p. 63].

Applying this to $P/\Psi^{n+\ell}$ gives

$$\int \left|\frac{P}{\Psi^{n+\ell}}\right|^p d\sigma^+ \leqslant C_2 N(\sigma^+) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{i\theta})\right|^p d\theta. \tag{16}$$

(VII) *Estimate the Integral Involving* $\sigma^{\#}$

Suppose that P has degree $v \leqslant n$. As $\Psi(z)/z$ has a finite non-zero limit as $z \to \infty$, $P(z)/\Psi(z)^v$ has a finite non-zero limit as $z \to \infty$. Then $h(t) := (P(\frac{1}{t})/\Psi(\frac{1}{t})^{n+\ell})$ has zeros in $|t| < 1$ corresponding only to zeros of $P(z)$ in $|z| > 1$ and a zero of multiplicity $n + \ell - v$ at $t = 0$, corresponding to the zero of $P(z)/\Psi(z)^{n+\ell}$ at $z = \infty$. Then we may apply Carleson's inequality (15) to h. The consequence is that

$$\int \left|\frac{P}{\Psi^{n+\ell}}\right|^p \left(\frac{1}{t}\right) d\sigma^{\#}(t) \leqslant C_2 N(\sigma^{\#}) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{-i\theta})\right|^p d\theta.$$

Combined with (14) and (16), this gives

$$\begin{aligned} &\int_\alpha^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta \\ &\qquad \leqslant C_0 C_1^p C_2 (N(\sigma^+) + N(\sigma^{\#})) \int_0^{2\pi} \left|\frac{P}{\Psi^{n+\ell}}(e^{i\theta})\right|^p d\theta. \end{aligned} \tag{17}$$

(VIII) *Pass from the Whole Unit Circle to* Δ *when* $p > 1$

Let Γ denote the whole unit circle, and let $|dt|$ denote arclength on Γ. Suppose that we have an estimate of the form

$$\int_{\Gamma \setminus \Delta} |g(t)|^p \, |dt| \leqslant C_3 \left(\int_\Delta |g(t)_+|^p \, |dt| + |g(t)_-|^p \, |dt| \right), \tag{18}$$

valid for all functions g analytic in $\mathbb{C} \setminus \Delta$, with limit 0 at ∞, and interior and exterior boundary values g_+ and g_- for which the right-hand side of (18) is finite. Here, C_3 depends only on p. (We shall establish such an inequality in the next step). We apply this to $g := P/\Psi^{n+\ell}$. Then as $\Psi_{\pm}$ have absolute value 1 on Δ, so that $|g_{\pm}| = |P|$ on Δ, we deduce that

$$\int_{\Gamma\setminus\Delta} |P(t)/\Psi(t)^{n+\ell}|^p \, |dt| \leqslant C_3 \int_\Delta |P(t)|^p \, |dt|$$

$$\Rightarrow \int_0^{2\pi} \left| \frac{P}{\Psi^{n+\ell}} (e^{i\theta}) \right|^p d\theta \leqslant \left(\int_\alpha^{2\pi-\alpha} |P(e^{i\theta})|^p \, d\theta \right) (1 + C_3).$$

Now (17) becomes

$$\int_\alpha^{2\pi-\alpha} |(P'\varepsilon)(e^{i\theta})|^p \, d\theta$$

$$\leqslant C_0 C_1^p C_2 (1 + C_3)(N(\sigma^+) + N(\sigma^\#)) \int_\alpha^{2\pi-\alpha} |P(e^{i\theta})|^p \, d\theta. \qquad (19)$$

(IX) *We Establish* (18) *for* $p > 1$

We note that inequalities like (18) are an essential ingredient of the procedure used in [8, 9] for proving weighted Markov–Bernstein inequalities, though there the unit ball was replaced by a half-plane. In the case $p = 2$, they were also used in [7]. We can follow the same procedure. Firstly we may use Cauchy's integral formula to deduce that

$$g(z) = \frac{1}{2\pi i} \int_\Delta \frac{g_-(t) - g_+(t)}{t - z} \, dt, \qquad z \notin \Delta.$$

Let χ denote the characteristic function of Δ and for functions $f \in L_1(\Delta)$, define the Hilbert transform on the unit circle,

$$H[f](z) := \frac{1}{i\pi} PV \int_\Gamma \frac{f(t)}{t - z} \, dt, \qquad \text{a.e.} \quad z \in \Gamma.$$

Here PV denotes Cauchy principal value. Then we see that for $z \in \Gamma \setminus \Delta$,

$$g(z) = \tfrac{1}{2} [H[\chi g_-](z) - H[\chi g_+](z)].$$

Now the Hilbert transform is a bounded operator on $L_p(\Gamma)$, that is

$$\int_\Gamma |H[f](t)|^p \, |dt| \leqslant C_4 \int_\Gamma |f(t)|^p \, |dt|,$$

where C_4 depends only on p [5]. We deduce that

$$\int_{\Gamma\backslash\Delta} |g(t)|^p \, |dt| \leqslant C_4 \left(\int_{\Delta} |g(t)_+|^p \, |dt| + |g(t)_-|^p \, |dt| \right),$$

so we have (18).

(X) *Pass from the Whole Unit Circle to* Δ *when* $p \leqslant 1$

We have to modify the previous procedure as the Hilbert transform is not a bounded operator on $L_p(\Gamma)$ when $p \leqslant 1$. It is only here that we really need the choice (10) of ℓ. Let

$$q := \ell p (>1).$$

Then we would like to apply (18) with p replaced by q and with

$$g := (P/\Psi^n)^{p/q} \, \Psi^{-1} = (P/\Psi^{n+\ell})^{p/q}. \tag{20}$$

The problem is that g does not in general possess the required properties. To circumvent this, we proceed as follows: firstly, we may assume that P has full degree n. For, if (3) holds when P has degree n, (and for every n) it also holds when P has degree $\leqslant n$, since ε_n is decreasing in n.

So assume that P has degree n. Then P/Ψ^n is analytic in $\mathbb{C}\backslash\Delta$ and has a finite non-zero limit at ∞, so is analytic at ∞. Now if all zeros of P lie on Δ, then we may define a single valued branch of g of (20) in $\overline{\mathbb{C}}\backslash\Delta$. Then (18) with q replacing p gives as before

$$\int_{\Gamma\backslash\Delta} |g(t)|^q \, |dt| \leqslant C_3 \left(\int_{\Delta} |g(t)_+|^q \, |dt| + |g(t)_-|^q \, |dt| \right)$$

$$\Rightarrow \int_{\Gamma\backslash\Delta} |P/\Psi^{n+\ell}|^p \, |dt| \leqslant 2C_3 \int_{\Delta} |P(t)|^p \, |dt|$$

and then we obtain an estimate similar to (19). When P has zeros in $\mathbb{C}\backslash\Delta$, we adopt a standard procedure to "reflect" these out of $\mathbb{C}\backslash\Delta$. Write

$$P(z) = d \prod_{j=1}^{n} (z - z_j).$$

For each factor $z - z_j$ in P with $z_j \notin \Delta$, we define

$$b_j(z) := \begin{cases} (z - z_j) \Big/ \left(\dfrac{\Psi(z) - \Psi(z_j)}{1 - \overline{\Psi(z_j)}\, \Psi(z)} \right), & z \neq z_j \\ (1 - |\Psi(z_j)|^2)/\Psi'(z_j), & z = z_j \end{cases}.$$

This is analytic in $\mathbb{C}\backslash\Delta$, does not have any zeros there, and moreover, since as $z \to \Delta$, $|\Psi(z)| \to 1$, we see that

$$|b_j(z)| = |z - z_j|, \quad z \in \Delta; \qquad |b_j(z)| \geqslant |z - z_j|, \quad z \in \mathbb{C}\backslash\Delta.$$

(Recall that we extended Ψ to Δ as an exterior boundary value). We may now choose a branch of

$$g(z) := \left[d\left(\prod_{z_j \notin \Delta} b_j(z) \right)\left(\prod_{z_j \in \Delta} (z - z_j) \right) \Big/ \Psi(z)^n \right]^{p/q} \Big/ \Psi(z)$$

that is single valued and analytic in $\mathbb{C}\backslash\Delta$, and has limit 0 at ∞. Then as $\Psi_{\pm}$ have absolute value 1 on Δ, so that $|g_{\pm}|^q = |P|^p$ on Δ, we deduce from (18) that

$$\begin{aligned} \int_{\Gamma\backslash\Delta} |P(t)/\Psi(t)^{n+\ell}|^p \, |dt| &\leqslant \int_{\Gamma\backslash\Delta} |g(t)|^q \, |dt| \\ &\leqslant C_3 \int_{\Delta} (|g(t)_+|^q + |g(t)_-|^q) \, |dt| \\ &= 2C_3 \int_{\Delta} |P(t)|^p \, |dt| \end{aligned}$$

and again we obtain an estimate similar to (19).

(XI) *Completion of the proof*

We shall show in Lemma 3.2 that

$$N(\sigma^+) + N(\sigma^\#) \leqslant C_4. \tag{21}$$

Then (19) becomes

$$\int_{\alpha}^{2\pi - \alpha} |(P'\varepsilon_n)(e^{i\theta})|^p \, d\theta \leqslant C_5 \int_{\alpha}^{2\pi - \alpha} |P(e^{i\theta})|^p \, d\theta.$$

So we have (3) with a constant C_5 that depends only on the numerical constants C_j, $1 \leqslant j \leqslant 4$ that arise from

(a) the bound on the conformal map Ψ;

(b) Carleson's inequality (15);

(c) the norm of the Hilbert transform as an operator on $L_p(\Gamma)$ and the choice of ℓ;

(d) the upper bound on the Carleson norms of σ^+ and $\sigma^\#$.

3. TECHNICAL ESTIMATES

Throughout we assume (4) to (6). We begin with some estimates on the function ε:

LEMMA 3.1. (a) *For $z, a \in \Delta$,*

$$|\varepsilon(z)-\varepsilon(a)| \leqslant 2\,|z-a|. \tag{22}$$

(b) *Let $0<K<\frac{1}{2}$. Then for $a, z \in \Delta$ such that $|z-a| \leqslant K\varepsilon(a)$, we have*

$$1-2K \leqslant \frac{\varepsilon(z)}{\varepsilon(a)} \leqslant 1+2K. \tag{23}$$

Proof. (a) Write $z=e^{i\theta}$; $a=e^{is}$. Now from (6),

$$|\varepsilon(z)-\varepsilon(a)| = \frac{1}{n}\left|\frac{\left[|R(z)|+\left(\frac{\pi-\alpha}{n}\right)^2\right]-\left[|R(a)|+\left(\frac{\pi-\alpha}{n}\right)^2\right]}{\left[|R(z)|+\left(\frac{\pi-\alpha}{n}\right)^2\right]^{1/2}+\left[|R(a)|+\left(\frac{\pi-\alpha}{n}\right)^2\right]^{1/2}}\right|$$

$$\leqslant \frac{|R(z)-R(a)|}{2(\pi-\alpha)}. \tag{24}$$

Here

$$R(a) = -4a\sin\left(\frac{s-\alpha}{2}\right)\sin\left(\frac{s+\alpha}{2}\right) = -4a\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{s}{2}\right),$$

so as

$$\frac{1}{\pi}(\pi-\alpha) \leqslant \cos\frac{\alpha}{2} = \sin\frac{\pi-\alpha}{2} \leqslant \frac{1}{2}(\pi-\alpha),$$

$$|R(a)| \leqslant 4\cos^2\frac{\alpha}{2} \leqslant (\pi-\alpha)^2.$$

Note that then also

$$\varepsilon(a) \leqslant \frac{\sqrt{2}}{n}(\pi-\alpha) \leqslant \frac{5}{n}\cos\frac{\alpha}{2}. \tag{25}$$

Next,

$$R(z)-R(a)=-4(z-a)\left(\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}\right)+4a\left(\cos^2\frac{\theta}{2}-\cos^2\frac{s}{2}\right),$$

so as $\theta\in[\alpha, 2\pi-\alpha]$,

$$|R(z)-R(a)|\leqslant 4\,|z-a|\cos^2\frac{\alpha}{2}+4\left|\sin\left(\frac{s-\theta}{2}\right)\sin\left(\frac{s+\theta}{2}\right)\right|.$$

Here

$$\begin{aligned}\left|\sin\left(\frac{s-\theta}{2}\right)\sin\left(\frac{s+\theta}{2}\right)\right| &\leqslant \left|\sin\left(\frac{s-\theta}{2}\right)\right|\left[\left|\sin\frac{s}{2}\cos\frac{\theta}{2}\right|+\left|\cos\frac{s}{2}\sin\frac{\theta}{2}\right|\right]\\ &\leqslant \left|\sin\left(\frac{s-\theta}{2}\right)\right|\left[2\cos\frac{\alpha}{2}\right]\\ &=|z-a|\cos\frac{\alpha}{2}.\end{aligned}$$

We have used the fact that both $s, \theta\in[\alpha, 2\pi-\alpha]$. So

$$|R(z)-R(a)|\leqslant 8\,|z-a|\cos\frac{\alpha}{2}.$$

Then (24) gives (22).

(b) This follows directly from (a). ∎

We next estimate the norms of the Carleson measures σ^+, $\sigma^{\#}$ defined by (9) and (12–13). Recall that the Carleson norm $N(\mu)$ of a measure μ with support in the unit ball is the least A such that

$$\mu(S)\leqslant Ah, \tag{26}$$

for every $0<h<1$ and for every sector

$$S:=\{re^{i\theta}: r\in[1-h, 1];\ |\theta-\theta_0|\leqslant h\}. \tag{27}$$

LEMMA 3.2. (a)

$$N(\sigma^+)\leqslant c_1. \tag{28}$$

(b)

$$N(\sigma^{\#})\leqslant c_2. \tag{29}$$

Proof. (a) We proceed much as in [7] or [8] or [9]. Let S be the sector (27) and let γ be a circle centre a, radius $\frac{\varepsilon(a)}{100}>0$. A necessary condition for γ to intersect S is that

$$|a-e^{i\theta_0}|\leqslant\frac{\varepsilon(a)}{100}+h.$$

(Note that each point of S that is on the unit circle is at most h in distance from $e^{i\theta_0}$.) Using Lemma 3.1(a), we continue this as

$$|a-e^{i\theta_0}|\leqslant\frac{\varepsilon(e^{i\theta_0})}{100}+\frac{2}{100}|a-e^{i\theta_0}|+h$$

$$\Rightarrow |a-e^{i\theta_0}|\leqslant\frac{\varepsilon(e^{i\theta_0})}{98}+2h=:\lambda \tag{30}$$

Next $\gamma\cap S$ consists of at most three arcs (draw a picture!) and as each such arc is convex, it has length at most $4h$. Therefore the total angular measure of $\gamma\cap S$ is at most $12h/(\varepsilon(a)/100)$. It also obviously does not exceed 2π. Thus if χ_S denote the characteristic function of S,

$$\int_{-\pi}^{\pi}\chi_S(a+\varepsilon(a)\,e^{i\theta})\,d\theta\leqslant\min\left\{2\pi,\frac{1200h}{\varepsilon(a)}\right\}.$$

Then from (9) and (12), we see that

$$\begin{aligned}\sigma^+(S)&\leqslant\sigma(S)\\ &\leqslant\int_{[\alpha,\,2\pi-\alpha]\cap\{s:\,|e^{is}-e^{i\theta_0}|\leqslant\lambda\}}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_S\left(e^{is}+\frac{\varepsilon(e^{is})}{100}\,e^{i\theta}\right)d\theta\right]ds\\ &\leqslant C_1\int_{[\alpha,\,2\pi-\alpha]\cap\{s:\,|e^{is}-e^{i\theta_0}|\leqslant\lambda\}}\min\left\{1,\frac{h}{\varepsilon(e^{is})}\right\}ds.\end{aligned} \tag{31}$$

We now consider two subcases:

(I) $h\leqslant\varepsilon(e^{i\theta_0})/100$

In this case,

$$\lambda<\frac{\varepsilon(e^{i\theta_0})}{25}<1,$$

recall (25) and (30). Then for s in the integral in (31),

$$|e^{is} - e^{i\theta_0}| \leqslant \lambda < 1$$

$$\Rightarrow 2\left|\sin\left(\frac{s-\theta_0}{2}\right)\right| = |e^{is} - e^{i\theta_0}| \leqslant \lambda < \frac{\varepsilon(e^{i\theta_0})}{25}$$

$$\Rightarrow |s-\theta_0|$$

and hence s, belongs to a set of linear measure at most $\leqslant C_2\varepsilon(e^{i\theta_0})$.

Also Lemma 3.1(b) gives

$$\varepsilon(e^{is}) \geqslant \tfrac{23}{25}\,\varepsilon(e^{i\theta_0}).$$

So (31) becomes

$$\sigma^+(S) \leqslant \sigma(S) \leqslant C_2\varepsilon(e^{i\theta_0})\,\frac{h}{\varepsilon(e^{i\theta_0})} = C_2 h.$$

(II) $h > \varepsilon(e^{i\theta_0})/100$

In this case $\lambda < 4h$. If $h < \frac{1}{4}$, we obtain $\lambda < 1$ so as above, for s in the integral in (31), $|s-\theta_0| < \pi$ and hence

$$2\left|\sin\left(\frac{s-\theta_0}{2}\right)\right| = |e^{is} - e^{i\theta_0}| \leqslant \lambda < 4h$$

$$\Rightarrow |s-\theta_0|$$

and hence s, belongs to a set of linear measure at most $C_2 h$.

Then (31) becomes

$$\sigma^+(S) \leqslant \sigma(S) \leqslant C_2 h \cdot 1 = C_2 h.$$

If $h > \frac{1}{4}$, it is easier to use

$$\sigma^+(S) \leqslant \sigma(S) \leqslant \sigma(\mathbb{C}) \leqslant C_1 2\pi \leqslant C_1 8\pi h.$$

In summary, we have proved that

$$N(\sigma^+) = \sup_{S,\,h} \frac{\sigma^+(S)}{h} \leqslant C_3,$$

where C_3 is independent of n, α, β. (It is also independent of p.)

(b) Recall that if S is the sector (27), then

$$\sigma^{\#}(S) = \sigma^-(1/S) \leqslant \sigma(1/S),$$

where

$$1/S = \left\{ re^{i\theta} \colon r \in \left[1, \frac{1}{1-h} \right]; |\theta + \theta_0| \leqslant h \right\}.$$

For small h, say for $h \in [0, 1/2]$, so that

$$\frac{1}{1-h} \leqslant 1 + 2h,$$

we see that exact same argument as in (a) gives

$$\sigma^{\#}(S) \leqslant \sigma(1/S) \leqslant C_4 h.$$

When $h \geqslant 1/2$, it is easier to use

$$\sigma^{\#}(S)/h \leqslant 2\sigma^{\#}(\mathbb{C}) \leqslant 2\sigma(\mathbb{C}) \leqslant C_5. \quad \blacksquare$$

4. THE PROOF OF THEOREM 1.1.

We deduce Theorem 1.1 from Theorem 1.2 as follows: if s_n is a trigonometric polynomial of degree $\leqslant n$, we may write

$$s_n(\theta) = e^{-in\theta} P(e^{i\theta}),$$

where P is an algebraic polynomial of degree $\leqslant 2n$. Then

$$|s_n'(\theta)| \; \varepsilon_{2n}(\varepsilon^{i\theta}) \leqslant n \; |P(e^{i\theta})| \; \varepsilon_{2n}(e^{i\theta}) + |P'(e^{i\theta})| \; \varepsilon_{2n}(\varepsilon^{i\theta}).$$

Moreover,

$$|e^{i\theta} - e^{i\alpha}| \; |e^{i\theta} - e^{i\beta}| = 4 \left| \sin\left(\frac{\theta - \alpha}{2}\right) \right| \left| \sin\left(\frac{\theta - \beta}{2}\right) \right|.$$

These last two relations and Theorem 1.2 easily imply (1). $\blacksquare$

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